# A Semantic Foundation for Gradual Set-theoretic Types

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The transition is **gradual**:

$$? \ \preccurlyeq \ ? 
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- In Semantic subtyping:

 $\begin{array}{l} \mbox{Types}\simeq\mbox{Sets of values}\\ \mbox{Subtyping}\simeq\mbox{Set-containment} \end{array}$ 

```
let map (condition : Bool) (f : \alpha \rightarrow \beta) (data : ) : =
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let map (condition : Bool) (f : α -> β) (data : ) : =
if condition then
List.map f data
else
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Runtime checks or casts are then inserted automatically by the compiler.

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let map (condition : Bool) (f : \alpha \rightarrow \beta) (data : ?) : ? =
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Runtime checks or casts are then inserted automatically by the compiler.

This is however very unsafe, as it accepts a string for example.

```
let map condition f
 (data : (α list ∨ α array) ) =
 if condition then
 List.map f data
else
 Array.map f data
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```
let map condition f
 (data : (α list ∨ α array) ∧ ?) =
 if condition then
  List.map f data
 else
  Array.map f data
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let map condition f
 (data : (α list ∨ α array) ∧ ?) =
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```
- By subtyping, (\alpha list \lor \alpha array) \land ? \leq ?.
```

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- By subtyping, ( $\alpha$  list  $\lor \alpha$  array)  $\land$  ?  $\leq$  ?.
- Can only be used with lists or arrays.
- No need for manual type checks.

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2. Embed this relation into typing rules.

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This gets even more complicated with set-theoretic types!

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**Important remark**: this translation is **only used** to define and compute relations, and **is not done in the source program**.

**Subtyping** only allows us to move **inside** the dynamic world, or **inside** the static world. It **does not** allow crossing the barrier. **Subtyping** only allows us to move **inside** the dynamic world, or **inside** the static world. It **does not** allow crossing the barrier.

As opposed to consistent subtyping, it is transitive:

? 
$$\leq$$
 ? ?  $\leq$  Int Int  $\leq$  ?

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As opposed to consistent subtyping, it is transitive:

$$? \leq ?$$
  $? \nleq Int$  Int  $\nleq ?$ 

It can be used to handle unions and intersections, by **simply plugging-in** the static version of **semantic subtyping**:

$$? \leq ? \lor Int$$
 Int  $\land ? \leq ?$ 

 $\begin{array}{ll} ? \preccurlyeq \tau & \text{for every } \tau \\ ? \rightarrow ? \preccurlyeq \tau_1 \rightarrow \tau_2 & \text{for every } \tau_1, \tau_2 \end{array}$ 

?  $\preccurlyeq \tau$  for every  $\tau$ ?  $\rightarrow$  ?  $\preccurlyeq \tau_1 \rightarrow \tau_2$  for every  $\tau_1, \tau_2$ 

And it is transitive:

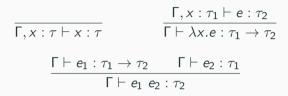
?  $\preccurlyeq$  ?  $\rightarrow$  ?  $\preccurlyeq$  ?  $\rightarrow$  Int  $\preccurlyeq$  Int  $\rightarrow$  Int

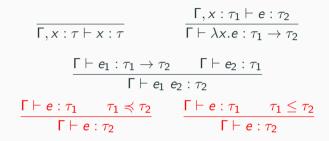
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?  $\preccurlyeq$  ?  $\rightarrow$  ?  $\preccurlyeq$  ?  $\rightarrow$  Int  $\preccurlyeq$  Int  $\rightarrow$  Int

Therefore it can be embedded into a type system as a **subsumption-like** rule: **materialization**.





$$\begin{array}{l} \overline{\Gamma}, x : \forall \vec{\alpha}. \tau \vdash x : \tau \{ \vec{\alpha} := \vec{t} \} \end{array} \qquad \begin{array}{l} \overline{\Gamma}, x : \tau_1 \vdash e : \tau_2 \\ \overline{\Gamma} \vdash \lambda x. e : \tau_1 \rightarrow \tau_2 \\ \hline \Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 & \Gamma \vdash e_2 : \tau_1 \\ \hline \Gamma \vdash e_1 \; e_2 : \tau_2 \end{array} \\ \\ \frac{\overline{\Gamma} \vdash e_1 : \tau_1 & \overline{\Gamma}, x : \operatorname{Gen}_{\Gamma}(\tau_1) \vdash e_2 : \tau}{\Gamma \vdash \operatorname{let} x = e_1 \; \operatorname{in} \; e_2 : \tau} \end{array}$$

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$$\frac{\overline{\Gamma}, x : \forall \vec{\alpha}. \tau \vdash x : \tau \{ \vec{\alpha} := \vec{t} \}}{\Gamma \vdash e_1 : \tau_1 \to \tau_2} \qquad \frac{\overline{\Gamma}, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x. e : \tau_1 \to \tau_2} \\
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And as a bonus, we get the static gradual guarantee for free!

```
For every type \tau \in \text{GTypes}, there exists t_1, t_2 \in \text{STypes} such that:

\tau \preccurlyeq t_1 \text{ and } \tau \preccurlyeq t_2

\forall \tau' \in \text{GTypes}. \ \tau \preccurlyeq \tau' \implies t_1 \leq \tau' \leq t_2
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We write  $t_1= au^{\Downarrow}$  and  $t_2= au^{\Uparrow}.$ 

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$$(? \rightarrow ?)^{\uparrow} = \mathbb{O} \rightarrow \mathbb{1} \qquad (? \rightarrow ?)^{\Downarrow} = \mathbb{1} \rightarrow \mathbb{O}$$

These types are computed in linear time!

We show the following:

$$\tau_1 \le \tau_2 \iff \begin{cases} \tau_1^{\Downarrow} \le \tau_2^{\Downarrow} \\ \tau_1^{\Uparrow} \le \tau_2^{\Uparrow} \end{cases}$$

We show the following:

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Moreover, we have that for every gradual type  $\tau$ ,

$$\tau \simeq \tau^{\Downarrow} \lor (? \land \tau^{\uparrow})$$

We show the following:

$$\tau_{1} \leq \tau_{2} \iff \begin{cases} \tau_{1}^{\downarrow} \leq \tau_{2}^{\downarrow} \\ \tau_{1}^{\uparrow} \leq \tau_{2}^{\uparrow} \end{cases} \qquad \tau_{1} \preccurlyeq \tau_{2} \iff \begin{cases} \tau_{1}^{\downarrow} \leq \tau_{2}^{\downarrow} \\ \tau_{2}^{\uparrow} \leq \tau_{1}^{\uparrow} \end{cases}$$

Moreover, we have that for every gradual type  $\tau$ ,

$$au \simeq au^{\Downarrow} \lor (? \land au^{\Uparrow})$$

We can use this representation to lift operators to gradual types!

$$\operatorname{dom}(\tau) \stackrel{\operatorname{def}}{=} \operatorname{dom}(\tau^{\uparrow}) \lor (? \land \operatorname{dom}(\tau^{\downarrow}))$$

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- 1. A simple method of **declaratively adding** gradual typing to any existing type system.
- 2. A **set-theoretic interpretation** of gradual types that has considerable consequences.
- 3. The algorithmic systems for our GTLC with set-theoretic types.
- 4. **Denotational semantics** for several calculi, including CDuce, and a GTLC without set-theoretic types.

- Fully unify our logical approach and our denotational semantics.

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